Cosmic Topology: a Brief Overview

M.J. Rebouças[*]

Centro Brasileiro de Pesquisas Físicas
Departamento de Relatividade e Partículas
Rua Dr. Xavier Sigaud, 150 ,
22290-180 Rio de Janeiro – RJ, Brazil

G.I. Gomero[†]

Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona, 145, 01405-900 São Paulo – SP, Brazil

(Dated: February 2, 2008)

Questions such as whether we live in a spatially finite universe, and what its shape and size may be, are among the fundamental open problems that high precision modern cosmology needs to resolve. These questions go beyond the scope of general relativity (GR), since as a (local) metrical theory GR leaves the global topology of the universe undetermined. Despite our present-day inability to *predict* the topology of the universe, given the wealth of increasingly accurate astro-cosmological observations it is expected that we should be able to *detect* it. An overview of basic features of cosmic topology, the main methods for its detection, and observational constraints on detectability are briefly presented. Recent theoretical and observational results related to cosmic topology are also discussed.

I. INTRODUCTION

Is the space where we live finite or infinite? The popular ancient Greek finite-world response, widely accepted in medieval Europe, is at a first sight open to a devastating objection: in being finite the world must have a limiting boundary. But this is impossible, because a boundary can only separate one part of the space from another: why not redefine the universe to include that other part? In this way a common-sense response to the above old cosmological question is that the universe has to be infinite otherwise something else would have to exist beyond its limits. This answer seems to be obvious and needing no further proof or explanation. However, in mathematics it is known that there are compact spaces (finite) with no boundary. They are called closed spaces. Therefore, our universe can well be spatially closed (topologically) with nothing else beyond its 'spatial limits'. This may be difficult to visualize because we are used to viewing from 'outside' objects which are embedded in our regular 3–dimensional space. But there is no need to exist any region beyond the spatial extent of the universe.

Of course, one might still ask what is outside such a closed universe. But the underlying assumption behind this question is that the ultimate physical reality is an infinite Euclidean space of some dimension, and nature needs not to adhere to this theoretical embedding framework. It is perfectly acceptable for our 3–space not to be embedded in any higher-dimensional space with no

physical grounds.

Whether the universe is spatially finite and what is its size and shape are among the fundamental open problems that high precision modern cosmology seeks to resolve. These questions of topological nature have become particularly topical, given the wealth of increasingly accurate astro-cosmological observations, especially the recent observations of the cosmic microwave background radiation (CMBR) [1]. An important point in the search for answers to these questions is that as a (local) metrical theory general relativity (GR) leaves the global topology of the universe undetermined. Despite this inability to *predict* the topology of the universe we should be able to devise strategies and methods to *detect* it by using data from astro-cosmological observations.

The aim of the article is to give a brief review of the main points on cosmic topology addressed in the talk delivered by one of us (MJR) in the XXIV Brazilian National Meeting on Particles and Fields, and discuss some recent results in the field. The outline of our paper is as follows. In section II we discuss how the cosmic topology issue arises in the context of the standard cosmology, and what are the main observational consequences of a non-trivial topology for the spatial section of the universe. In section III we review the two main statistical methods to detect cosmic topology from the distribution of discrete cosmic sources. In section IV we describe the search for circles in the sky, an important method which has been devised for the detection of cosmic topology from CMBR. In section V we discuss the detectability of cosmic topology and present examples on how one can decide whether a given topology is detectable or not according to recent observations. Finally, in section VI we briefly discuss recent results on cosmic topology, and present some con-

---

[*]Electronic address: reboucas@cbpf.br
[†]Electronic address: german@ift.unesp.br

cluding remarks.

II. NONTRIVIAL TOPOLOGY AND PHYSICAL CONSEQUENCES

The isotropic expansion of the universe, the primordial abundance of light elements and the nearly uniform cosmic microwave background radiation constitute the main observational pillars for the standard cosmological model, which provides a very successful description of the universe. Within the framework of standard cosmology, the universe is described by a space-time manifold $\mathcal{M}_4 = \mathbb{R} \times M$ endowed with the homogeneous and isotropic Robertson-Walker (RW) metric

$$ds^2 = -c^2 dt^2 + R^2(t)\left\{\, d\chi^2 + f^2(\chi)\left[\, d\theta^2 + \sin^2\theta\, d\phi^2\,\right]\,\right\},\tag{1}$$

where t is a cosmic time, $f(\chi) = (\chi, \sin\chi, \sinh\chi)$ depending on the sign of the constant spatial curvature $k = (0, 1, -1)$, and $R(t)$ is the scale factor. The spatial section M is often taken to be one of the following (simply-connected) spaces: Euclidean $\mathbb{E}^3$, spherical $\mathbb{S}^3$, or hyperbolic space $\mathbb{H}^3$. This has led to a common misconception that the Gaussian curvature k of M is all one needs to decide whether this 3–space is finite or not. However, the 3-space M may equally well be one of the possible quotient manifolds $M = \widetilde{M}/\Gamma$, where Γ is a discrete and fixed-point free group of isometries of the corresponding covering space $\widetilde{M} = (\mathbb{E}^3, \mathbb{S}^3, \mathbb{H}^3)$. Quotient manifolds are multiply connected: compact in three independent directions with no boundary (closed), or compact in two or at least one independent direction. The action of Γ tessellates $\widetilde{M}$ into identical cells or domains which are copies of what is known as fundamental polyhedron (FP). In forming the quotient manifolds M the essential point is that they are obtained from $\widetilde{M}$ by identifying points which are equivalent under the action of the discrete group Γ. Hence, each point on the quotient manifold M represents all the equivalent points on the covering manifold $\widetilde{M}$. A simple example of quotient manifold in two dimensions is the 2–torus $T^2 = S^1 \times S^1 = \mathbb{E}^2/\Gamma$. The covering space clearly is $\mathbb{E}^2$, and a FP is a rectangle with opposite sides identified. This FP tiles the covering space $\mathbb{E}^2$. The group Γ consists of discrete translations associated with the side identifications.

In a multiply connected space any two points can always be joined by more than one geodesic. Since the radiation emitted by cosmic sources follows geodesics, the immediate observational consequence of a spatially closed universe is that light from distant objects can reach a given observer along more than one path — the sky may show multiple images of radiating sources [cosmic objects or cosmic microwave background radiation from the last scattering surface - (LSS)]. Clearly we are assuming here that the radiation (light) must have sufficient time to reach the observer at $p \in M$ (say) from multiple directions, or put in another way, that the universe is suf-

ficiently small so that this repetitions can be observed. In this case the observable horizon χ_{hor} exceeds at least the smallest characteristic size of M at p [55], and the topology of the universe is in principle detectable.

A question that arises at this point is whether one can use the topological multiple images of the same celestial objects such as cluster of galaxies, for example, to determine a nontrivial cosmic topology [56]. Besides the pioneering work by Ellis [2], others including Sokolov and Shvartsman [3], Fang and Sato [4], Starobinskii [5], Gott [6] and Fagundes [7] and Fagundes and Wichoski [8], used this feature in connection with closed flat and non-flat universes. It has been recently shown that the topology of a closed flat universe can be reconstructed with the observation of a very small number of multiple images [9].

In practice, however, the identification of multiple images is a formidable observational task to carry out because it involves a number of problems, some of which are:

- Images are seen from different angles (directions), which makes it very hard to recognize them as identical due to morphological effects;

- High obscuration regions or some other object can mask or even hide the images;

- Two images of a given cosmic object at different distances correspond to different periods of its life, and so they are in different stages of their evolutions, rendering problematic their identification as multiple images.

These difficulties make clear that a direct search for multiples images is not overly promising, at least with available present-day technology. On the other hand, they motivate new search strategies and methods to determine (or just detect) the cosmic topology from observations. In the next section we shall discuss statistical methods, which have been devised to determine a possible nontrivial topology of the universe from the distribution of discrete cosmic sources.

III. PAIR SEPARATIONS STATISTICAL METHODS

On the one hand the most fundamental consequence of a multiply connected spatial section M for the universe is the existence of multiple images of cosmic sources, on the other hand a number of observational problems render the direct identification of these images practically impossible. In the statistical approaches we shall discuss in this section instead of focusing on the direct recognition of multiple images, one treats statistically the images of a given cosmic source, and use (statistical) indicators or signatures in the search for a sign of a nontrivial topology. Hence the statistical methods are not plagued by direct

recognition difficulties such as morphological effects, and distinct stages of the evolution of cosmic sources.

The key point of these methods is that in a universe with detectable nontrivial topology at least one of the characteristic sizes of the space section M is smaller than a given survey depth χ_{obs}, so the sky should show multiple images of sources, whose 3–D positions are correlated by the isometries of the covering group Γ. These methods rely on the fact that the correlations among the positions of these images can be couched in terms of distance correlations between the images, and use statistical indicators to find out signs of a possible nontrivial topology of M.

In 1996 Lehoucq *et al.* [10] proposed the first statistical method (often referred to as cosmic crystallography), which looks for these correlations by using pair separations histograms (PSH). To build a PSH we simply evaluate a suitable one-to-one function F of the distance d between a pair of images in a catalogue $\mathcal{C}$, and define $F(d)$ as the pair separation: $s = F(d)$. Then we depict the number of pairs whose separation lie within certain sub-intervals J_i partitions of $(0, s_{max}]$, where $s_{max} = F(2\chi_{max})$, and χ_{max} is the survey depth of $\mathcal{C}$. A PSH is just a normalized plot of this counting. In most applications in the literature the separation is taken to be simply the distance between the pair $s = d$ or its square $s = d^2$, J_i being, respectively, a partition of $(0, 2\chi_{max}]$ and $(0, 4\chi_{max}^2]$.

The PSH building procedure can be formalized as follows. Consider a catalogue $\mathcal{C}$ with n cosmic sources and denote by $\eta(s)$ the number of pairs of sources whose separation is s. Divide the interval $(0, s_{max}]$ in m equal sub-intervals (bins) of length $\delta s = s_{max}/m$, being

$$J_i = \left(s_i - \frac{\delta s}{2}, \, s_i + \frac{\delta s}{2}\right], \,; \quad i = 1, 2, \ldots, m \, ,$$

and centered at $s_i = (i - \frac{1}{2})\delta s$. The PSH is defined as the following counting function:

$$\Phi(s_i) = \frac{2}{n(n-1)} \, \frac{1}{\delta s} \sum_{s \in J_i} \eta(s) \, , \qquad (2)$$

which can be seen to be subject to the normalization condition $\sum_{i=1}^{m} \Phi(s_i) \, \delta s = 1$. An important advantage of using *normalized* PSH's is that one can compare histograms built up from catalogues with different number of sources.

An example of PSH obtained through simulation for a universe with nontrivial topology is given in Fig. 1. Two important features should be noticed: (i) the presence of the very sharp peaks (called spikes); and (ii) the existence of a 'mean curve' above which the spikes stands. This curve corresponds to an expected pair separation histogram (EPSH) $\Phi_{exp}(s_i)$, which is a typical PSH from which the statistical noise has been withdrawn, that is $\Phi_{exp}(s_i) = \Phi(s_i) - \rho(s_i)$, where $\rho(s_i)$ represents the statistical fluctuation that arises in the PSH $\Phi(s_i)$.

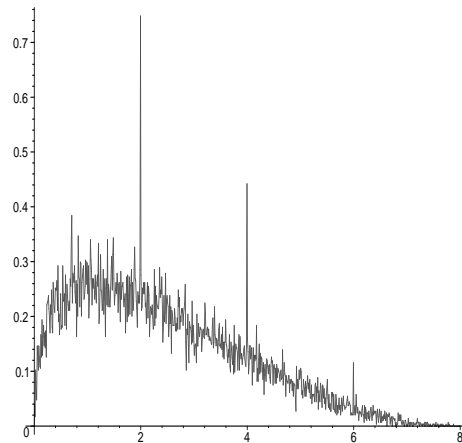The primary expectation was that the distance correlations would manifest as topological spikes in PSH's,

FIG. 1: Typical PSH for a flat universe with a 3–torus topology. The horizontal axis gives the squared pair separation s^2, while the vertical axis provides a normalized number of pairs.

and that the spike spectrum of topological origin would be a definite signature of the topology [10]. While the first simulations carried out for specific flat manifolds appeared to confirm this expectation [10], histograms subsequently generated for specific hyperbolic manifolds revealed that the corresponding PSH's exhibit no spikes [11, 12]. Concomitantly, a theoretical statistical analysis of the distance correlations in PSH's was accomplished, and a proof was presented that the spikes of topological origin in PSH's are due to just one type of isometry: the Clifford translations (CT) [13], which are isometries $g_t \in \Gamma$ such that for all $p \in \widetilde{M}$ the distance $d(p, g_t p)$ is a constant (see also in this regard [11]). Clearly the CT's reduce to the regular translations in the Euclidean spaces (for more details and simulations see [14, 15, 16]). Since there is no CT translation in hyperbolic geometry this result explains the absence of spikes in the PSH's of hyperbolic universes with nontrivial detectable topology. On the other hand, it also makes clear that distinct manifolds which admit the same Clifford translations in their covering groups present the same spike spectrum of topological origin. Therefore the topological spikes are not sufficient for unambiguously determine the topology of the universe.

In spite of these limitations, the most striking evidence of multiply-connectedness in PSH's is indeed the presence of topological spikes, which result from translational isometries $g_t \in \Gamma$. It was demonstrated [13, 14] that the other isometries g manifest as very tiny deformations of the expected pair separation histogram $\Phi_{exp}^{sc}(s_i)$ corresponding to the underlying simply connected universe [17, 18]. Furthermore, in PSH's of universes with nontrivial topology the amplitude of the sign of nontranslational isometries was shown to be smaller than the statistical noise [14], making clear that one cannot use PSH to reveal these isometries.

In brief, the only significant (measurable) sign of a nontrivial topology in PSH are the spikes, but they can

be used merely to disclose (not to determine) a possible nontrivial topology of universes that admit Clifford translations: any flat, some spherical, and no hyperbolic universes.

The impossibility of using the PSH method for the detection of the topology of hyperbolic universes motivated the development of a new scheme called *collecting correlated pairs method* (CCP method) [19] to search for cosmic topology.

In the CCP method it is used the basic feature of the isometries, i.e., that they preserve the distances between pairs of images. Thus, if (p, q) is a pair of arbitrary images (correlated or not) in a given catalogue $\mathcal{C}$, then for each $g \in \Gamma$ such that the pair (gp, gq) is also in $\mathcal{C}$ we obviously have

$$d(p, q) = d(gp, gq) . \tag{3}$$

This means that for a given (arbitrary) pair (p, q) of images in $\mathcal{C}$, if there are n isometries $g \in \Gamma$ such that both images gp and gq are still in $\mathcal{C}$, then the separation $s(p, q)$ will occur n times.

The easiest way to understand the CCP method is by looking into its computer-aimed procedure steps, and then examine the consequences of having a multiply connected universe with detectable topology. To this end, let $\mathcal{C}$ be a catalogue with n sources, so that one has $P = n(n-1)/2$ pairs of sources. The CCP procedure consists on the following steps:

1. Compute the P separations $s(p, q)$, where p and q are two images in the catalogue $\mathcal{C}$;

2. Order the P separations in a list $\{s_i\}_{1 \leq i \leq P}$ such that $s_i \leq s_{i+1}$;

3. Create a list of *increments* $\{\Delta_i\}_{1 \leq i \leq P-1}$, where $\Delta_i = s_{i+1} - s_i$;.

4. Define the CCP index as

$$\mathcal{R} = \frac{\mathcal{N}}{P-1},$$

where $\mathcal{N} = Card\{i \, : \, \Delta_i = 0\}$ is the number of times the increment is null.

If the smallest characteristic length of M exceeds the survey depth ($r_{inj} > \chi_{obs}$) the probability that two pairs of images are separated by the same distance is zero, so $\mathcal{R} \approx 0$. On the other hand, in a universe with detectable nontrivial topology ($\chi_{obs} > r_{inj}$) given $g \in \Gamma$, if p and q as well as gp and gq are images in $\mathcal{C}$, then: (i) the pairs (p, q) and (gp, gq) are separated by the same distance; and (ii) when Γ admits a translation g_t the pairs $(p, g_t p)$ and $(q, g_t q)$ are also separated by the same distance. It follows that when a nontrivial topology is detectable, and a given catalogue $\mathcal{C}$ contains multiple images, then $\mathcal{R} > 0$, so the CCP index is an indicator of a detectable nontrivial topology of the spatial section M of the universe. Note that although $\mathcal{R} > 0$ can be used as a sign of multiply

connectedness, it gives no indication as to what the actual topology of M is. Clearly whether one can find out that M is multiply connected (compact in at least one direction) is undoubtedly a very important step, though.

In more realistic situations, uncertainties in the determination of positions and separations of images of cosmic sources are dealt with through the following extension of the CCP index:

$$\mathcal{R}_\epsilon = \frac{\mathcal{N}_\epsilon}{P-1} \, ,$$

where $\mathcal{N}_\epsilon = Card\{i \, : \, \Delta_i \leq \epsilon\}$, and $\epsilon > 0$ is a parameter that quantifies the uncertainties in the determination of the pairs separations.

Both PSH and CCP statistical methods rely on the accurate knowledge of the three-dimensional positions of the cosmic sources. The determination of these positions, however, involves inevitable uncertainties, which basically arises from: (i) uncertainties in the determination of the values of the cosmological density parameters Ω_{m0} and $\Omega_{\Lambda 0}$; (ii) uncertainties in the determination of both the red-shifts (due to spectroscopic limitations), and the angular positions of cosmic objects (displacement, due to gravitational lensing by large scale objects, e.g.); and (iii) uncertainties due to the peculiar velocities of cosmic sources, which introduce peculiar red-shift corrections. Furthermore, in most studies related to these methods the catalogues are taken to be complete, but real catalogues are incomplete: objects are missing due to selection rules, and also most surveys are not full sky coverage surveys. Another very important point to be considered regarding these statistical methods is that most of cosmic objects do not have very long lifetimes, so there may not even exist images of a given source at large red-shift. This poses the important problem of what is the suitable source (candle) to be used in these methods.

Some of the above uncertainties, problems and limits of the statistical methods have been discussed by Lehoucq *et al.* [20], but the robustness of these methods still deserves further investigation. So, for example, a quantitative study of the sensitivity of spikes and CCP index with respect to the uncertainties in the positions of the cosmic sources, which arise from unavoidable uncertainties in values of the density parameters is being carried out [21]. In [21] it is also determined the optimal values of the bin size (in the PSH method) and the ϵ parameter (in the CCP method) so that the correlated pairs are collected in a way that the topological sign is preserved.

For completeness we mention that Bernui [22] has worked with a similar method which uses angular pair separation histogram (APSH) in connection with CMBR.

To close this section we refer the reader to references [23, 24], which present variant statistical methods (see also the review article [25]).

IV. CIRCLES IN THE SKY

The deepest surveys currently available are the CMBR temperature anisotropy maps with $z_{LSS} \approx 10^3$. Thus, given the current high quality and resolution of such maps, the most promising searches for cosmic topology through multiple images of radiating sources are based on pattern repetitions of these CMBR anisotropies.

The last scattering surface (LSS) is a sphere of radius χ_{LSS} on the universal covering manifold of the comoving space at present time. If a nontrivial topology of space is detectable, then this sphere intersects some of its topological images. Since the intersection of two spheres is a circle, then CMBR temperature anisotropy maps will have matched circles, i.e. pairs of equal radii circles (centered on different point on the LSS sphere) that have the same pattern of temperature variations [26].

These matched circles will exist in CMBR anisotropy maps of universes with any detectable nontrivial topology, regardless of its geometry. Thus in principle the search for 'circles in the sky' can be performed without any *a priori* information (or assumption) on the geometry, and on the topology of the universe.

The mapping from the last scattering surface to the night sky sphere is a conformal map. Since conformal maps preserves angles, the identified circle at the LSS would appear as identified circles on the night sky sphere. A pair of matched circles is described as a point in a six-dimensional parameter space. These parameters are the centers of each circle, which are two points on the unit sphere (four parameters), the angular radius of both circles (one parameter), and the relative phase between them (one parameter).

Pairs of matched circles may be hidden in the CMBR maps if the universe has a detectable topology. Therefore to observationally probe nontrivial topology on the available largest scale, one needs a statistical approach to scan all-sky CMBR maps in order to draw the correlated circles out of them. To this end, let $\mathbf{n}_1 = (\theta_1, \varphi_1)$ and $\mathbf{n}_2 = (\theta_2, \varphi_2)$ be the center of two circles C_1 and C_2 with angular radius ν. The search for the matching circles can be performed by computing the following correlation function [26]:

$$S(\alpha) = \frac{\langle 2T_1(\pm\phi)T_2(\phi+\alpha)\rangle}{\langle T_1(\pm\phi)^2 + T_2(\phi+\alpha)^2\rangle} \,, \qquad (4)$$

where T_1 and T_2 are the temperature anisotropies along each circle, α is the relative phase between the two circles, and the mean is taken over the circle parameter ϕ: $\langle \ \rangle = \int_0^{2\pi} d\phi$. The plus $(+)$ and minus $(-)$ signs in (4) correspond to circles correlated, respectively, by non-orientable and orientable isometries.

For a pair of circles correlated by an isometry (perfectly matched) one has $T_1(\pm\phi) = T_2(\phi + \alpha_*)$ for some α_*, which gives $S(\alpha_*) = 1$, otherwise the circles are uncorrelated and so $S(\alpha) \approx 0$. Thus a peaked correlation function around some α_* would mean that two matched circles, with centers at $\mathbf{n}_1$ and $\mathbf{n}_2$, and angular radius ν, have been detected.

From the above discussion it is clear that a full search for matched circles requires the computation of $S(\alpha)$, for any permitted α, sweeping the parameter sub-space $(\theta_1, \varphi_1, \theta_2, \varphi_2, \nu)$, and so it is indeed computationally expensive. Nevertheless, such a search is currently in progress, and preliminary results using the first year WMAP data indicate the lack of antipodal, and nearly antipodal, matched circles with radii larger than $25°$ [27]. Here nearly antipodal means circles whose center are separated by more than $170°$.

According to these first results, the possibility that our universe has a torus-type *local shape* is discarded, i.e. any flat topology with translations smaller than the diameter of the sphere of last scattering is ruled out. As a matter of fact, as they stand these preliminary results exclude any topology whose isometries produce antipodal images of the observer, as for example the Poincaré dodecahedron model [28], or any other homogeneous spherical space with detectable isometries.

Furthermore, since detectable topologies (isometries) do not produce, in general, antipodal correlated circles, a little more can be inferred from the lack or nearly antipodal matched circles. Thus, in a flat universe, e.g., any screw motion may generate pairs of circles that are not even nearly antipodal, provided that the observer's position is far enough from the axis of rotation [29]. As a consequence, our universe can still have a flat topology, other than the 3-torus, but in this case the axis of rotation of the screw motion corresponding to a pair of matched circles would pass far from our position. Similar results also hold for spherical universes with non-translational isometries generating pairs of matched circles. Indeed, the universe could have the topology of, e.g., an inhomogeneous lens space $L(p, q)$, but with both equators of minimal injectivity radius passing far from us [57]. These points also make clear the crucial importance of the position of the observer relative to the 'axis of rotation' in the matching circles search scheme for inhomogeneous spaces (in this regard see also [30]).

To conclude, 'circles in the sky' is a promising method in the search for the topology of the universe, and may provide more general and realistic constraints on the shape and size of our universe in the near future. An important point in this regard is the lack of computational less expensive search for matched circles, which can be archived by restricting (in the light of observations) the expected detectable isometries, confining therefore the parameter space of realistic search for correlated circles as indicated, for example, by Mota *et al.* [31].

V. DETECTABILITY OF COSMIC TOPOLOGY

In the previous sections we have assumed that the topology of the universe is detectable, and focussed our attention on strategies and methods to discover or even

determine a possible nontrivial topology of the universe. In this section we shall examine the consequences of this underlying detectability assumption in the light of the current astro-cosmological observations which indicate that our universe is nearly flat ($\Omega_0 \approx 1$) [32]. Although this near flatness of the universe does not preclude a non-trivial topology it may push the smallest characteristic size of M to a value larger than the observable horizon χ_{hor}, making it difficult or even impossible to detect by using multiple images of radiating sources (discrete cosmic objects or CMBR maps). The extent to which a nontrivial topology may or may not be detected has been examined in locally flat [33], spherical [34, 35, 36] or hyperbolic [36, 37, 38, 39] universes. The discussion below is based upon our articles [34, 36, 37, 38], so we shall focus on nearly (but not exactly) flat universes (for a study of detectability of flat topology see [33]).

The study of the detectability of a possible nontrivial topology of the spatial sections M requires topological typical scale which can be put into correspondence with observation survey depths. A suitable characteristic size of M is the so-called injectivity radius $r_{inj}(x)$ at $x \in M$, which is defined in terms of the length of the smallest closed geodesics that pass through x as follows.

A closed geodesic that passes through a point x in a multiply connected manifold M is a segment of a geodesic in the covering space $\widetilde{M}$ that joins two images of x. Since any such pair of images are related by an isometry $g \in \Gamma$, the length of the closed geodesic associated to any fixed isometry g, and that passes through x, is given by the corresponding distance function

$$\delta_g(x) \equiv d(x, gx) . \tag{5}$$

The injectivity radius at x then is defined by

$$r_{inj}(x) = \frac{1}{2} \min_{g \in \widetilde{\Gamma}} \{ \delta_g(x) \} , \tag{6}$$

where $\widetilde{\Gamma}$ denotes the covering group without the identity map. Clearly, a sphere with radius $r < r_{inj}(x)$ and centered at x lies inside a fundamental polyhedron of M.

For a specific survey depth χ_{obs} a topology is said to be undetectable by an observer at a point x if $\chi_{obs} < r_{inj}(x)$, since in this case every image catalogued in the survey lies inside the fundamental polyhedron of M centered at the observer's position x. In other words, there are no multiple images in the survey of depth χ_{obs}, and therefore any method for the search of cosmic topology based on their existence will not work. If, otherwise, $\chi_{obs} > r_{inj}(x)$, then the topology is potentially detectable (or detectable in principle).

In a globally homogeneous manifold, the distance function for any covering isometry g is constant. Therefore, the injectivity radius is constant throughout the whole space, and so if the topology is potentially detectable (or undetectable) by an observer at x, it is detectable (or undetectable) by any other observer at any other point in the same space. However, in globally inhomogeneous manifolds the injectivity radius varies from point to point, thus in general the detectability of cosmic topology depends on both the observer's position x and survey depth. Nevertheless, for globally inhomogeneous manifolds one can define the global injectivity radius by

$$r_{inj} = \min_{x \in M} \{ r_{inj}(x) \} , \tag{7}$$

and state an 'absolute' undetectability condition. Indeed, for a specific survey depth χ_{obs} a topology is undetectable by any observer (located at any point x) in the space provided that $r_{inj} > \chi_{obs}$.

Incidentally, we note that for globally inhomogeneous manifolds one can define the so-called injectivity profile $\mathcal{P}(r)$ of a manifold as the probability density that a point $x \in M$ has injectivity radius $r_{inj}(x) = r$. The quantity $\mathcal{P}(r)dr$ clearly provides the probability that $r_{inj}(x)$ lies between r and $r + dr$, and so the injectivity profile curve is essentially a histogram depicting how much of a manifold's volume has a given injectivity radius (for more detail on this point see Weeks [39]). An important point is that the injectivity profile for non-flat manifolds of constant curvature is a topological invariant since these manifolds are rigid.

In order to apply the above detectability of cosmic topology condition in the context of standard cosmology, we note that in non-flat RW metrics (1), the scale factor $R(t)$ is identified with the curvature radius of the spatial section of the universe at time t, and thus χ can be interpreted as the distance of any point with coordinates (χ, θ, ϕ) to the origin (in the covering space) in units of curvature radius, which is a natural unit of length.

To illustrate now the above condition for detectability (undetectability) of cosmic topology, in the light of recent observations [1, 32] we assume that the matter content of the universe is well approximated by dust of density ρ_m plus a cosmological constant Λ. In this cosmological setting the curvature radius R_0 of the spatial section is related to the total density parameter Ω_0 through the equation

$$R_0^2 = \frac{kc^2}{H_0^2(\Omega_0 - 1)} , \tag{8}$$

where H_0 is the Hubble constant, k is the normalized spatial curvature of the RW metric (1), and where here and in what follow the subscript 0 denotes evaluation at present time t_0. Furthermore, in this context the redshift-distance relation in units of the curvature radius, $R_0 = R(t_0)$, reduces to

$$\chi(z) = \sqrt{|1 - \Omega_0|} \int_1^{1+z} \frac{dx}{\sqrt{x^3 \Omega_{m0} + x^2(1 - \Omega_0) + \Omega_{\Lambda 0}}} , \tag{9}$$

where Ω_{m0} and $\Omega_{\Lambda 0}$ are, respectively, the matter and the cosmological density parameters, and $\Omega_0 \equiv \Omega_{m0} + \Omega_{\Lambda 0}$. For simplicity, on the left hand side of (9) and in many places in the remainder of this article, we have left

implicit the dependence of the function χ on the density components.

A first qualitative estimate of the constraints on detectability of cosmic topology from nearflatness can be obtained from the function $\chi(\Omega_{m0}, \Omega_{\Lambda0}, z)$ given by (9) for a fixed survey depth z. Figure 2 clearly demonstrates the rapid way χ drops to zero in a narrow neighbourhood of the $\Omega_0 = 1$ line. This can be understood intuitively from (8), since the natural unit of length (the curvature radius R_0) goes to infinity as $\Omega_0 \to 1$, and therefore the depth χ (for any fixed z) of the observable universe becomes smaller in this limit. From the observational point of view, this shows that the detection of the topology of the nearly flat universes becomes more and more difficult as $\Omega_0 \to 1$, a limiting value favoured by recent observations. As a consequence, by using any method which relies on observations of repeated patterns the topology of an increasing number of nearly flat universes becomes undetectable in the light of the recent observations, which indicate that $\Omega_0 \approx 1$.
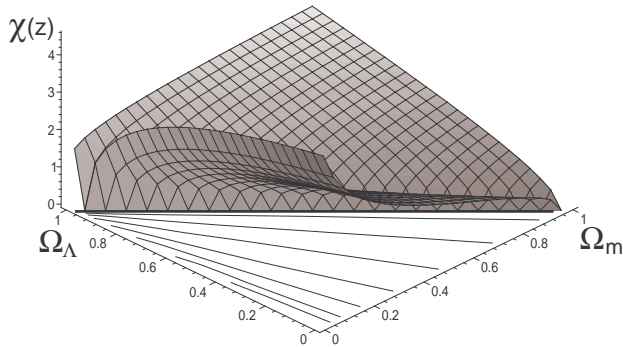

FIG. 2: The behaviour of $\chi(\Omega_{m0}, \Omega_{\Lambda0}, z)$ for a fixed $z = 1100$ as a function of the density parameters $\Omega_{\Lambda0}$ and Ω_{m0}.

From the above discussion it is clear that cosmic topology may be undetectable for a given survey up to a depth z_{max}, but detectable if one uses a deeper survey. At present the deepest survey available corresponds to $z_{max} = z_{LSS} \approx 10^3$, with associated depth $\chi(z_{LSS})$. So the most promising searches for cosmic topology through multiple images of radiating sources are based on CMBR.

To quantitatively illustrate the above features of the detectability problem, we shall examine the detectability of cosmic topology of the first ten smallest (volume) hyperbolic universes.

To this end we shall take the following interval of the density parameters values consistent with current observations: $\Omega_0 \in [0.99, 1)$ and $\Omega_{\Lambda0} \in [0.63, 0.73]$. In this hyperbolic sub-interval one can calculate the largest value of $\chi_{obs}(\Omega_{m0}, \Omega_{\Lambda0}, z)$ for the last scattering surface ($z = 1100$), and compare with the injectivity radii r_{inj} to decide upon detectability. From (9) one obtains $\chi_{obs}^{max} = 0.337$.

Table I summarizes our results which have been refined upon and reconfirmed by Weeks [39]. It makes explicit that there are undetectable topologies even if one uses

$Manifold$	r_{inj}	CMBR
m003(-3,1)	0.292	—
m003(-2,3)	0.289	—
m007(3,1)	0.416	U
m003(-4,3)	0.287	—
m004(6,1)	0.240	—
m004(1,2)	0.183	—
m009(4,1)	0.397	U
m003(-3,4)	0.182	—
m003(-4,1)	0.176	—
m004(3,2)	0.181	—

TABLE I: Restrictions on detectability of cosmic topology for $\Omega_0 = 0.99$ with $\Omega_{\Lambda0} \in [0.63, 0.73]$ for the first ten smallest known hyperbolic manifolds. Here U stands for undetectable topology with CBMR ($z_{max} = 1100$), while the dash denotes detectable in principle.

CMBR.

We note that similar results hold for spherical universes with values of the density parameters within the current observational bounds (for details see [34, 35, 36]). This makes apparent that there exist nearly flat hyperbolic and spherical universes with undetectable topologies for $\Omega_0 \approx 1$ favoured by recent observations.

The most important outcome of the results discussed in this section is that, as indicated by recent observations (and suggested by inflationary scenarios) Ω_0 is close (or very close) to one, then there are both spherical and hyperbolic universe whose topologies are undetectable. This motivates the development of new strategies and/or methods in the search for the topology of nearly flat universes, perhaps based on the local physical effect of a possible nontrivial topology. In this regard see [40, 41, 42, 43, 44, 45], for example.

VI. RECENT RESULTS AND CONCLUDING REMARKS

In this section we shall briefly discuss some recent results and advances in the search for the shape of the universe, which have not been treated in the previous sections. We also point out some problems, which we understand as important to be satisfactorily dealt with in order to make further progress in cosmic topology.

One of the intriguing results from the analysis of WMAP data is the considerably low value of the CMBR quadrupole and octopole moments, compared with that predicted by the infinite flat ΛCDM model. Another noteworthy feature is that, according to WMAP data analysis by Tegmark et al., both the quadrupole and the octopole moments have a common preferred spatial axis along which the power is suppressed [58].

This alignment of the low multipole moments has been suggested as an indication of a direction along which a possible shortest closed geodesics (characteristic of multiply connected spaces) of the universe may be [50]. Moti-

vated by this as well as the above anomalies, test using S-statistics [51] and matched circles furnished no evidence of a nontrivial topology with diametrically opposed pairs of correlated circles [50]. It should be noticed, however, that these results do no rule out most multiply connected universe models because S-statistics is a method sensitive only to Euclidean translations, while the search for circles in the sky, which is, in principle, appropriate to detect any topology, was performed in a limited three-parameter version, which again is only suitable to detect translations.

At a theoretical level, although strongly motivated by high precision data from WMAP, it has been shown that if a *very* nearly flat universe has a detectable nontrivial topology, then it will exhibit the generic local shape of (topologically) $\mathbb{R}^2 \times \mathbb{S}^1$[59], irrespective of its global shape [31]. In this case, from WMAP and SDSS the data analysis, which indicates that $\Omega_0 \approx 1$ [32], one has that if the universe has a detectable topology, it is very likely that it has a preferred direction, which in turn is in agreement with the observed alignement of the quadrupole and octopole moments of the CMBR anisotropies. In this context, it is relevant to check whether a similar alignment of higher order multipole ($\ell > 3$) takes place in order to reinforce a possible nontrivial local shape of our 3–space. In this connection it is worth mentioning that Hajian and Souradeep [52, 53] have recently suggested a set of indicators κ_ℓ ($\ell = 1, 2, 3, ...$) which for non-zero values indicate and quantify statistical anisotropy in a CMBR map. Although κ_ℓ can be potentially used to discriminate between different cosmic topology candidates, they give no information about the directions along which the isotropy may be violated, and therefore other indicators should be devised to extract anisotropy directions from CBMR maps.

In ref. [31] it has also been shown that in a very nearly flat universe with detectable nontrivial topology, the observable (detectable) isometries will behave nearly like translations. Perhaps if one use Euclidean space to locally approximate a nearly flat universe with detectable topology, the detectable isometries can be approximated by Euclidean isometries, and since these isometries are not translations, they have to be screw motions. As a consequence, an approximate local shape of a nearly flat universe with detectable topology would look like a *twisted* cylinder, i.e. a flat manifold whose covering group is generated by a screw motion. Work toward a proof of this conjecture is being carried out by our research group.

Before closing this overview we mention that the study of the topological signature (possibly) encoded in CMBR maps as well as to what extent the cosmic topology CMBR detection methods are robust against distinct observational effects such as, e.g., Suchs-Wolfe and the thickness of the LSS effects, will benefit greatly from accurate simulations of these maps in the context of the FLRW models with multiply connected spatial sections. A first step in this direction has been achieved by Riazuelo *et al.* [54], with special emphasis on the effect of the topology in the suppression of the low multipole moments. Along this line it is worth studying through computer-aided simulations the effect of a nontrivial cosmic topology on the nearly alignments of the quadrupole and the octopole moments (spatial axis along which the power is suppressed).

To conclude, cosmic topology is at present a very active research area with a number of important problems, ranging from how the characterization of the local shape of the universe may observationally be encoded in CMBR maps, to the development of more efficient computationally searches for matching circles, taking into account possible restrictions on the detectable isometries, and thereby confining the parameter space which realistic 'circles in the sky' searches need to concentrate on. It is also of considerable interest the search for the statistical anisotropy one can expect from a universe with nontrivial space topology. Finally, it is important not to forget that there are almost flat (spherical and hyperbolic) universes, whose spatial topologies are undetectable in the light of current observations with the available methods, and our universe can well have one of such topologies. In this case we have to devise new methods and strategies to detect the topology of the universe.

Acknowledgments

[1] C.L. Bennett et al. , Astrophys. J. **583**, 1 (2003); D.N. Spergel et al. , Astrophys. J. Suppl. **148**, 175 (2003); G. Hinshaw et al. , Astrophys. J. Suppl. **148**, 135 (2003); C.L. Bennett et al. , Astrophys. J. Suppl. **148**, 1 (2003).

[2] G.F.R. Ellis, Gen. Rel. Grav. **2**, 7 (1971).

[3] D.D. Sokolov and V.F. Shvartsman, Sov. Phys. JETP **39**, 196 (1974).

[4] L-Z. Fang and Sato, Gen. Rel. Grav. **17** 1117 (1985).

[5] D.D. Sokolov and A.A. Starobinsky, Sov. Astron. **19**, 629 (1975).

[6] J.R. Gott, Mon. Not. R. Astron. Soc. **193**, 153 (1980).

[7] H.V. Fagundes, Phys. Rev. Lett. **51**, 417 (1983).

[8] H.V. Fagundes and U.F. Wichoski, Astrophys. J. **322**, L52 (1987).

[9] G.I. Gomero, Class. Quantum Grav. **20**, 4775 (2003).

[10] R. Lehoucq, M. Lachièze-Rey and J.-P. Luminet, Astron. Astrophys. **313**, 339 (1996).

[11] R. Lehoucq, J.-P. Luminet and J.-P. Uzan, Astron. Astrophys. **344**, 735 (1999).

[12] H.V. Fagundes and E. Gausmann, *Cosmic Crystallogra-*

phy in Compact Hyperbolic Universes, astro-ph/9811368 (1998).

[13] G.I. Gomero, A.F.F. Teixeira, M.J. Rebouças and A. Bernui, Int. J. Mod. Phys. D **11**, 869 (2002). Also gr-qc/9811038 (1998).

[14] G.I. Gomero, M.J. Rebouças and A.F.F. Teixeira, Phys. Lett. A **275**, 355 (2000).

[15] G.I. Gomero, M.J. Rebouças and A.F.F. Teixeira, Int. J. Mod. Phys. D **9**, 687 (2000).

[16] G.I. Gomero, M.J. Rebouças and A.F.F. Teixeira, Class. Quantum Grav. **18**, 1885 (2001).

[17] A. Bernui and A.F.F. Teixeira, *Cosmic crystallography: three multi-purpose functions*, gr-qc/9904180.

[18] M.J. Rebouças, Int. J. Mod. Phys. D **9**, 561 (2000).

[19] J.-P. Uzan, R. Lehoucq and J.-P. Luminet, Astron. Astrophys. **351**, 766 (1999).

[20] R. Lehoucq, J.-P. Uzan and J.-P. Luminet, Astron. Astrophys. **363**, 1 (2000).

[21] A. Bernui, G.I. Gomero, B. Mota and M.J. Rebouças, in preparation (2004). A preliminar version was presented by M.J. Rebouças in the section COT1: Topology of the Universe, of the *X Marcel Grossmann Meeting* (July 20 – 26, 2003).

[22] A. Bernui, private communication.

[23] H.V. Fagundes and E. Gausmann, Phys. Lett. A **261** 235 (1999).

[24] B.F. Roukema and A. Edge, Mon. Not. R. Astron. Soc. **292**, 105 (1997).

[25] M. Lachièze-Rey and J.-P. Luminet, Phys. Rep. **254**, 135 (1995) ;
J.J. Levin, Phys. Rep. **365**, 251 (2002).

[26] N.J. Cornish, D. Spergel and G. Starkman, Class. Quantum Grav. **15**, 2657 (1998).

[27] N.J. Cornish, D.N. Spergel, G.D. Starkman and E. Komatsu, *Constraining the Topology of the Universe*, astro-ph/0310233.

[28] J.-P. Luminet, J. Weeks, A. Riazuelo, R. Lehoucq and J.-P. Uzan, Nature **425**, 593 (2003).

[29] G.I. Gomero, *'Circles in the Sky' in twisted cylinders*, astro-ph/0310749.

[30] A. Riazuelo, J. Weeks, J.-P. Uzan and J.-P. Luminet, *Cosmic microwave background anisotropies in multi-connected flat spaces*, astro-ph/0311314.

[31] B. Mota, G.I. Gomero, M.J. Reboucas and R. Tavakol, *What do very nearly flat detectable cosmic topologies look like?*, astro-ph/0309371.

[32] M. Tegmark et al., *Cosmological parameters from SDSS and WMAP*, astro-ph/0310723.

[33] G.I. Gomero and M.J. Rebouças, Phys. Lett. A **311**, 319 (2003).

[34] G.I. Gomero, M.J. Rebouças and R. Tavakol, Class. Quantum Grav. **18**, 4461 (2001).

[35] J. Weeks, R. Lehoucq, J.-P. Uzan, Class. Quant. Grav. **20**, 1529 (2003).

[36] B. Mota, M.J. Rebouças and R. Tavakol, Class. Quantum Grav. **20**, 4837 (2003).

[37] G.I. Gomero, M.J. Rebouças and R. Tavakol, Class. Quantum Grav. **18**, L145 (2001).

[38] G.I. Gomero, M.J. Rebouças and R. Tavakol, Int. J. Mod. Phys. A **17**, 4261 (2002).

[39] J.R. Weeks, Mod. Phys. Lett. A **18**, 2099 (2003).

[40] W. Oliveira, M.J. Rebouças and A.F.F. Teixeira, Phys. Lett. A **188**, 125 (1994).

[41] A. Bernui, G.I. Gomero, M.J. Rebouças and A.F.F. Teixeira, Phys. Rev. D **57**, 4699 (1998).

[42] M.J. Rebouças, R. Tavakol and A.F.F. Teixeira, Gen. Rel. Grav. **30**, 535 (1998).

[43] G.I. Gomero, M.J. Rebouças, A.F.F. Teixeira and A. Bernui, Int. J. Mod. Phys. A **15**, 4141 (2000).

[44] D. Muller, H.V. Fagundes and R. Opher, Phys. Rev. D **63**, 123508 (2001).

[45] D. Muller, H.V. Fagundes and R. Opher Phys. Rev. D **66**, 083507 (2002).

[46] M. Tegmark, A. de Oliveira–Costa and A.J.S. Hamilton, *A high resolution foreground cleaned CMB map from WMAP*, astro-ph/0302496.

[47] L.Z. Fang & M. Houjun, Mod. Phys. Lett. A **2**, 229 (1987).

[48] I.Yu. Sokolov, JETP Lett. **57**, 617 (1993).

[49] A.A. Starobinsky, JETP Lett. **57**, 622 (1993).

[50] A. de Oliveira–Costa, M. Tegmark, M. Zaldarriaga and A.J.S. Hamilton, *The significance of the largest scale CMB fluctuations in WMAP*, astro-ph/0307282.

[51] A. de Oliveira–Costa, G.F. Smoot & A.A. Starobinsky, Astrophys. J. **468**, 457 (1996).

[52] A. Hajian and T. Souradeep, *Statistical Isotropy of CMB and Cosmic Topology*, astro-ph/0301590.

[53] A. Hajian and T. Souradeep, *Measuring Statistical Isotropy of the CMB Anisotropy*, astro-ph/0308001.

[54] A. Riazuelo, J.-P. Uzan, R. Lehoucq and J. Weeks, *Simulating Cosmic Microwave Background maps in multi-connected spaces*, astro-ph/0212223.

[55] This is the so-called *injectivity radius* $r_{inj}(p)$. A more detailed discussion on this point will be given in section V.

[56] There are basically three types of catalogues which can possibly be used in the search for multiple images in the universe: clusters of galaxies, with redshifts up to $z_{max} \approx$ 0.3; active galactic nuclei with a redshift cut-off of $z_{max} \approx$ 4; and maps of the CMBR with a redshift of $z \approx 10^3$.

[57] In spherical geometry, the equators of minimal injectivity radius of an orientable non-translational isometry correspond to the axis of rotation of an Euclidean screw motion [31].

[58] Incidentally, it was the fitting to the observed low values of the quadrupole and the octopole moments of the CMB temperature fluctuations that motivated Poincaré dodecahedron space topology [28], which according to 'cirlces in the sky' plus WMAP analysis is excluded [27]. Nevertheless, the Poincaré dodecahedron space proposal was an important step in cosmic topology to the extent that for the first time a possible nontrivial cosmic topology was tested against accurate CMBR data.

[59] Or more rarely $\mathbb{R} \times \mathbb{T}^2$.